LECTURE 10 DIFFERENTIATION RULES

In the previous lecture, we dealt with the derivative as a function. For a given function $f(x)$, we have written down two equivalent definitions of the derivative, as a function of x, namely,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{z\to x}\frac{f(z)-f(x)}{z-x},$$

should either limit exists.

In real world problems, the form of $f(x)$ is not always known but rather written as various forms, such as $f(x) = ax^n$ or as a product of functions $f(x) = g(x)h(x)$, and many other forms.

Constants

The simplest form of $f(x)$ is a constant function, $f(x) = c$. Then,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

Thus, the derivative of any constant function is 0.

Polynomials

Example. Newton's free fall. From physics, you learned that the distance travelled by a free falling object in vacuum is given by

$$h(t) = \frac{1}{2}gt^2$$

where t is the time the object spends in air. The question is now to calculate your instantaneous speed $v(t)$ of falling at each time.

We understand that speed is distance over time, namely, an average rate of change,

$$\bar{v}(t,h) = \frac{h(t+h)-h(t)}{h}$$

Thus,

$$v(t) = \lim_{h\to 0}\bar{v}(t,h) = \lim_{h\to 0}\frac{h(t+h)-h(t)}{h} = \frac{1}{2}g\lim_{h\to 0}\frac{(t+h)^2-t^2}{h} = gt$$

as expected, velocity is acceleration times time.

In general,

$$\frac{d}{dx}x^n = nx^{n-1}$$

for <u>any</u> real number n. If $n = 0$, $x^0 = 1$ is a constant, whose derivative is 0 (where this formula also works).

Proof.

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{z\to x}\frac{z^n-x^n}{z-x}\\
&= \lim_{z\to x}\frac{(z-x)\left(z^{n-1}+z^{n-2}x+\cdots+zx^{n-2}+x^{n-1}\right)}{z-x}\\
&= \lim_{z\to x}\left(z^{n-1}+z^{n-2}x+\cdots+zx^{n-2}+x^{n-1}\right)\\
&= x^{n-1}+x^{n-1}+\cdots+x^{n-1}+x^{n-1}, \quad n \text{ terms}\\
&= nx^{n-1}
\end{aligned}$$

□

Example. Consider $f(x) = \frac{1}{\sqrt[3]{x^4}}$. Find $f'(x)$.

Solution. You must realise that you can always convert radical expressions to powers. First, deal with the radical.

$$\sqrt[3]{x^4} = x^{\frac{4}{3}}$$

and therefore

$$f(x) = x^{-\frac{4}{3}}.$$

By power rule,

$$f'(x) = -\frac{4}{3}x^{-\frac{4}{3}-1} = -\frac{4}{3}x^{-\frac{7}{3}}.$$

Constant Multiple of A Function

The next form of $f(x)$ is just a constant multiple of a function, say $f(x) = cg(x)$. Then,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{cg(x+h)-cg(x)}{h} \\ &= c\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= cg'(x), \end{aligned}$$

which implies that you can always factor out the constant multiplier.

Sums (and Differences)

Then we deal with derivatives of functions formed by sums, differences, and later products and quotients (the four basic operations.

$$\begin{aligned} \frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x)+g'(x) \end{aligned}$$

Products

Example. Let $W(t)$ be the number of workers at a factory at time t. Let $E(t)$ be the number of toys produced per worker per hour at time t. Now, $W'(t)$ is how fast workers are entering or exiting the factory, since it could well depend on the time of the day, such as different lunch hours, even traffic in the morning. $E'(t)$ is how fast the efficiency of the factory per worker is changing, which is again, also affected by the time of the day, such as higher temperature in the afternoon that worsens working condition.

Then, the product, $W(t)E(t)$, is the total rate at which the factory produces toys, that is, the number of toys produced by the factory per hour – we can call it the factory's output per hour. A factory examiner may wonder, is this output increasing or decreasing? The answer lies in $\frac{d}{dt}(W(t)E(t))$, the rate at which the factory's output per hour is changing.

What should $\frac{d}{dt}(W(t)E(t))$ depend on? Certainly $W'(t)$, if $E(t)$ is constant, then how fast the factory gains or loses workers should affect its output rate. Also, if $W(t)$ is constant, then the efficiency of each worker $E'(t)$ should also matter. At the same time, suppose we have a small workforce, while each worker has increasing efficiency. The effect on the overall factory output won't matter so much (as opposed to a large workforce, compare Monaco vs. China, say). Lastly, we say $E(t)$ is also important – if it is very small, even with an increasing inflow of workers who effectively don't do much, the increase in total output rate of the factory is still small.

The above argument means, $\frac{d}{dt}(W(t)E(t))$ should scale positively with $W(t)$, $E(t)$, $W'(t)$ and $E'(t)$, namely, every increase in each term should increase $\frac{d}{dt}(W(t)E(t))$. Now, in which way?

$\times$	W	ΔW
E		$E\Delta W$
ΔE	$W\Delta E$	$\Delta W\Delta E$

In more proper calculus terms, consider

$$\begin{aligned}\Delta W &= W(t+h) - W(t),\\ \Delta E &= E(t+h) - E(t).\end{aligned}$$

Now, given two differentiable functions $W(t)$ and $E(t)$, we want to find $(WE)'(t)$. We prove it with a geometric argument. By definition of the derivative,

$$\begin{aligned}(WE)'(t) &= \frac{d}{dt}\left((WE)(t)\right)\\ &= \lim_{h\to 0}\frac{(WE)(t+h) - (WE)(t)}{h}\\ &= \lim_{h\to 0}\frac{W(t+h)E(t+h) - W(t)E(t)}{h}\\ &= \lim_{h\to 0}\frac{E(t)\Delta W + W(t)\Delta E + \Delta W\Delta E}{h}\\ &= \lim_{h\to 0}\frac{E(t)\Delta W}{h} + \lim_{h\to 0}\frac{W(t)\Delta E}{h} + \lim_{h\to 0}\frac{\Delta W\Delta E}{h}\\ &= E(t)W'(t) + W(t)E'(t) + \left(\lim_{h\to 0}\frac{(W(t+h) - W(t))}{h}\right)\left(\cancelto{0}{\lim_{h\to 0} E(t+h) - E(t)}\right)\\ &= E(t)W'(t) + W(t)E'(t) + W'(t)\cdot 0\\ &= E(t)W'(t) + W(t)E'(t)\end{aligned}$$

You note that the total length of each side is $W(t) + \Delta W = W(t+h)$ and $E(t+h)$ respectively. The numerator after the 3rd equality is the total area of the rectangle minus that of the top left one. You then obtain the sum of the remaining three rectangles.

Example. Find the derivative of

(1) $f(x) = \left(3 - x^2\right)\left(x^3 - x + 1\right)$

Solution. Easy with product rule, though you can expand into a polynomial and do power rule.

$$\begin{aligned}f'(x) &= \left[\frac{d}{dx}\left(3 - x^2\right)\right]\left(x^3 - x + 1\right) + \left(3 - x^2\right)\left[\frac{d}{dx}\left(x^3 - x + 1\right)\right]\\ &= (-2x)\left(x^3 - x + 1\right) + \left(3 - x^2\right)\left(3x^2 - 1\right)\\ &= -2x^4 + 2x^2 - 2x + 9x^2 - 9x^4 - 3 + x^2\\ &= -11x^4 + 12x^2 - 2x - 3\end{aligned}$$

(2) $f(z) = \left(\frac{1+3z}{3z}\right)(3 - z)$

Solution. Simplify first. No need to use the quotient rule which we technically haven't introduced.

$$\begin{aligned} f'(z) &= \left[\frac{d}{dz}\left(\frac{1+3z}{3z}\right)\right](3-z) + \left(\frac{1+3z}{3z}\right)\left[\frac{d}{dz}(3-z)\right] \\ &= \left[\frac{d}{dz}\left(\frac{1}{3z}+1\right)\right](3-z) + \left(\frac{1+3z}{3z}\right)(-1) \\ &= \left(-\frac{1}{3z^2}\right)(3-z) - \frac{1+3z}{3z} \\ &= \frac{z-3}{3z^2} - \frac{1+3z}{3z} \\ &= \frac{z-3-z-9z^2}{3z^2} \\ &= -\frac{3+9z^2}{3z^2} \\ &= -\frac{1+3z^2}{z^2} \end{aligned}$$

Quotient

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example. Find the derivative of $g(x) = \frac{x^2-4}{x+\frac{1}{2}}$.

Solution. Direct application.

$$\begin{aligned} g'(x) &= \frac{\left(x+\frac{1}{2}\right)\frac{d}{dx}\left(x^2-4\right) - \left(x^2-4\right)\frac{d}{dx}\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2} \\ &= \frac{\left(x+\frac{1}{2}\right)(2x) - \left(x^2-4\right)}{\left(x+\frac{1}{2}\right)^2} \end{aligned}$$

Higher Order Derivatives

Example. Find the n^{th} derivative of $f(x) = \frac{1}{x}$.

Solution.

$$\begin{aligned} f'(x) &= -\frac{1}{x^2} \\ f''(x) &= \frac{2}{x^3} \\ f^{(3)}(x) &= -\frac{3\cdot 2\cdot 1}{x^4} \\ f^{(4)}(x) &= \frac{4\cdot 3\cdot 2\cdot 1}{x^5} \\ &\vdots \\ f^{(n)}(x) &= \frac{(-1)^n\, n!}{x^{n+1}} \end{aligned}$$